Lect24-0415 Htpy

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Let X, Y be spaces. Two continuous maps $f, g : X \longrightarrow Y$ are homotopic if $\exists$ continuous $H : X \times [0,1] \longrightarrow Y$, call homotopy

such that
$$\left.\begin{aligned} H(x,0) &= f(x) \\ H(x,1) &= g(x) \end{aligned}\right\} \quad \forall\ x \in X$$

Notation. $f \simeq g$ or $f \overset{H}{\simeq} g$

Example

① A rotation R_α on $\mathbb{R}^2$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto R_\alpha(x) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Any two R_α, R_β are homotopic

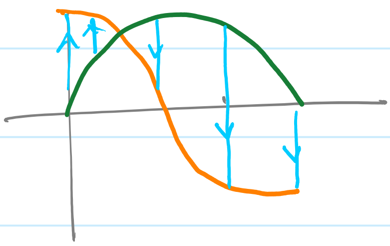
e.g $\quad H(x,t) = R_{(1-t)\alpha + t\beta}(x)$

Clearly, homotopy may not be unique.

② $f, g : [0, \pi] \longrightarrow \mathbb{R}^2$

$f(x) = \sin x$

$g(x) = \cos x$



$$H(x,t) = -\sin\left(\frac{t\pi}{2} - x\right)$$

Htpy Examples

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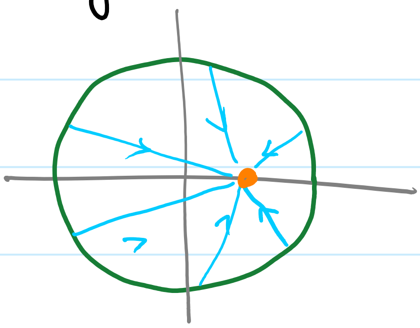
③ $X = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$, $Y = \mathbb{C} = \mathbb{R}^2$

$f, g : S^1 \longrightarrow \mathbb{C}$, $f(z) = z$, $g(z) = \frac{1}{2}$



$H(z,t) = (1-t)z + \frac{t}{2}$

④ $X = S^1$, $Y = \mathbb{R}^2 \setminus \{(0,0)\}$

For the above f, g, $f \not\simeq g$.

Null homotopic

A map $c : X \longrightarrow Y$ with $c(x) = y_0$ $\forall x \in X$

is called a constant map (onto $y_0 \in Y$)

If $f : X \longrightarrow Y$ satisfies $f \simeq c$ then

f is null homotopic or homotopically trivial.

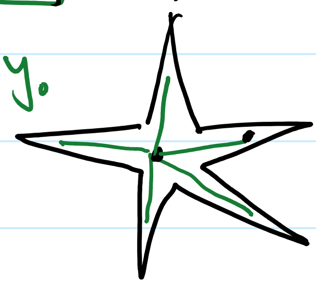
Fact. Any map $f : X \longrightarrow \mathbb{R}^n$, $n \geq 1$, is

null homotopic.

$H : X \times [0,1] \longrightarrow \mathbb{R}^n$, $H(x,t) = (1-t)f(x)$

Example $Y \subset \mathbb{R}^n$ is called star-shaped if

$\exists y_0 \in Y \; \forall y \in Y \; \underbrace{\{(1-t)y + t y_0 : t \in [0,1]\}}_{\text{straight line joining } y \text{ to } y_0} \subset Y$



Qu. Can we replace the straight

lines by other continuous paths?

Punctured plane

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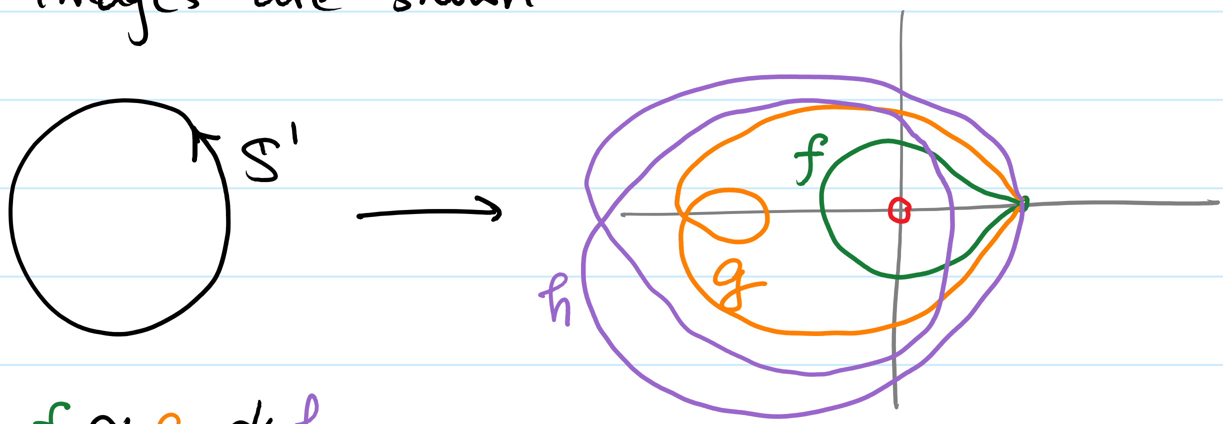
Example. Consider the following three maps $f, g, h : S^1 \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$. Their images are shown



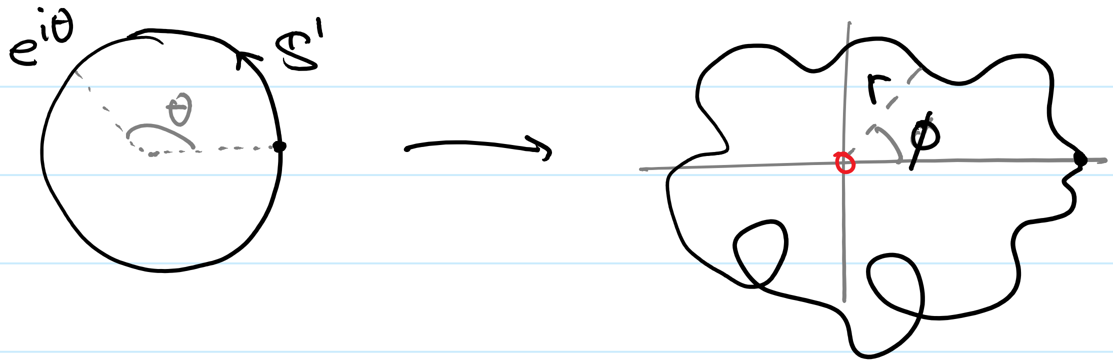
$f \simeq g \not\simeq h$

This can only be understood intuitively now.

Intuition

For any map $S^1 \longrightarrow \mathbb{R}^2 \setminus \{(0,0)\}$, it can be expressed as $e^{i\theta} \longmapsto re^{i\phi}$ where

$$r = r(\theta) > 0 \quad \text{and} \quad \phi = \phi(\theta)$$



We only need to worry about ϕ because any two $r_1, r_2 > 0$ can be easily homotopic.

Argument

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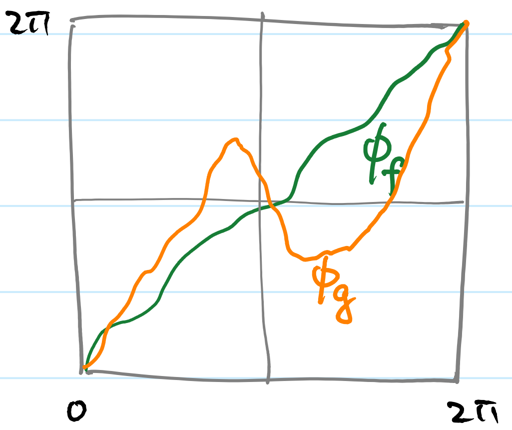
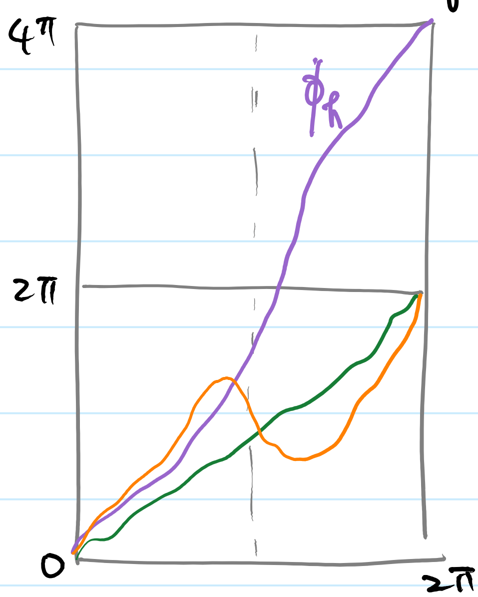
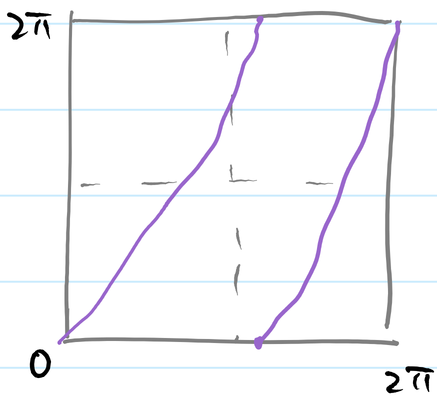
Without loss of generality, assume $\phi(0)=0$.
Then, when one varies θ in the domain S^1,
$\phi=\phi(\theta)$ changes dependantly continuously.
For the example of f and g, the graphs of ϕ are drawn below



Note that the two "ends" at $(0,0)$ and $(2\pi,2\pi)$ actually correspond to the same point on the loops of f and g.

In the above pictures, it is easy to continuously change ϕ_f to ϕ_g with the two end-points fixed.
This gives a homotopy between f and g.
However, the graph of h is different.





Branch of argument

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If we expect ϕ_h goes from $(0,0)$ to $(2\pi, 2\pi)$, we can only have the discontinuous graph shown on the right hand side picture.

To have a continuous ϕ_h, the graph goes from $(0,0)$ to $(2\pi, 4\pi)$.

One cannot at the same time fixed the end-points and continuously change ϕ_h to any of ϕ_f or ϕ_g.

Winding number In any cases, for the pictures of f, g, h, we can define winding numbers, which is an invariant.

$$w(f) = w(g) = \frac{2\pi}{2\pi} = 1 \quad \text{and}$$

$$w(h) = \frac{4\pi}{2\pi} = 2$$

As $w(f) = w(g) \neq w(h)$,

$$f, g \not\simeq h$$